Finite-Dimensional Hopf Algebras

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Abstract:

TThis paper mainly discussed various characterizations for the finite-dimensional Hopf algebras over algebraically closed field and has characteristic 0. And further showed that the order of antipode of the Hopf algebras is finite, but also provides a hint on how to estimate the order of the antipodes.

Keywords:

Finite-dimensional Hopf algebras; Order of the antipode ; Trace; Semisimple; Cosemisimple; Eigenvalue; Distinguished grouplike

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1 Introduction

Throughout this paper is a algebraically closed field and has characteristic0, is a finite -dimensional $K-Hopf$ algebra with antipode which is a diagonalizable operator and C is a $K-$coalgebra . There is a convenient adaptation of the Heyneman–Sweedler[1] singma notation for coalgebras and comodules as $\Delta(c)=\sum c_{(1)}\otimes c_{(2)}$ and $\rho(c)=\sum c_{(-1)}\otimes c_{(0)}\quad \forall c\in C$.

Definition1.1[1]

A grouplike elements of C is a $c\in C$ which satisfies the following conditions: $\Delta(c)=c\otimes c$ and $\varepsilon(c)=1,$ the set of C grouplike elements of is denoted $G(C)$.

We firstly recall the following actions as module structures:

(1) H^* is a left $H-$ module via $(h\rightarrow h^*)(g)=h^*(gh)$ for $h,g\in H,h^*\in H^*$.

(2) H^* is a rigt $H-$ module via for $h,g\in H,h^*\in H^*.$

(3) H is a left h H^*- module via $h^*\rightarrow h=\sum h^*(h_2)h_1$ for $h^*\in H^*,h\in H.$

(4) H is a right H^*- module via $h\leftarrow h^*=\sum h^*(h_1)h_2$ for $h^*\in H^*,h\in H.$

If $g\in H$ is a grouplike element as in Definition1.1, we can denote by

$$L_g=\{m\in H^*\big|h^*m=h^*(g)m \text{ for any } h^*\in H^*\}$$

and

$$R_g=\{n\in H^*\big|nh^*=h^*(g)n \text{ for any } h^*\in H^*\} \text{for any } h^*\in H^*\}$$

which are ideals of H^* and $L_1=\int_l$, $R_1=\int_r$. Also recall from[1] that L_g and R_g are 1-dimensional,and there exists a grouplike element d such that $R_d=L_1$, where d is called the distinguished grouplike. We can perform the same constructions on the dual algebra H^* . More precisely,

for any $\eta\in G(H^*)=A\lg(H,K)$ we can define

$$L_\eta=\{x\in H\big|hx=\eta(h)x \text{ for any } h\in H\}$$

$$R_\eta=\{y\in H\big|yh=\eta(h)y \text{ for any } h\in H\}\ .$$

We remark that if we keep the same definition we gave for L_g, then L_η should be a subspace of H^{**}.The set L_η, as defined above ,is just the preimage of this subspace via the canonical Isomorphism $\theta:H\rightarrow H^{**}$.From the above it follows that the subspaces L_η and R_η are ideals

of dimension 1 in H, and there exists $\alpha\in G(H^*)$ such that $R_\alpha=L_\varepsilon$. This element α is the distinguished grouplike element in H^*.

Remark1.2[1]

If H is semisimple and cosemisimple, then distinguished grouplike in H and H^* are equal to 1 and ε , respectively.

Lemma1.3[2]

Suppose that H is a Hopf algebra over K . Then

(1) The only subspaces of H which are both a left ideal and left coideal of H are and H

(2) If H contains a non-zero finite -dimensional left or right ideal. Then H is finite-dimensional.

Lemma1.4[3]

Let C be a finite-dimensional coalgebra over K. Then $U \mapsto U^{\perp}$ is a one-one inclusion reversing correspondence between the set of coideals (respectively subcoalgebras, left coideals, right coideals) of C and the set of subalgebras (respectively ideals, left ideals, right ideals) of the dual algebra C^*.

Lemma1.5[3]

If $C_n(K)$ is a simple coalgebra over K for all $n \geq 1$.Then any simple coalgebra over K is isomorphic to $C_n(K)$ for some $n \geq 1$.

Lemma1.6[4]

Suppose $U$and V be vector spaces over K and $F: V^* \to U^*$ is the transpose of a linear map $f: U \to V$. If J and I are subspaces of V^* and U^* respectively. Then $F(J) \subseteq I$ implies $f(I^{\perp}) \subseteq J^{\perp}$.

Remark1.7[4]

For a subspace $V$of U let $res_V^U: U^* \to V^*$ be the restriction map which is thus defined by $res_V^U(u^*) = u^*|V$ for all $u^* \in U^*$. Notice that $Ker(res_V^U) = V^{\perp}$. Hence $U^*/V^{\perp} \cong V^*$ as vector spaces. Therefore we have the formula $Dim(U^*/V^{\perp}) = Dim(V^*)$. In particular $V^{\perp}$ is a cofinite subspace of U^* if and only if V is a finite-dimensional subspace of U. Also notice that $res_V^U = i^*$, where $i: V \to U$ is the inclusion map.

Definition1.8[4]

For $a \in H$, $a^* \in H^*$,$b \in H$, define endomorphisms $L(a^*)$ and $R(a^*)$ in $End(H)$ by $End(H)$ by $L(a^*)(b) = a^* \to b$ and $R(a^*)(b) = b \leftarrow a^*$, on the other hand, $l(a)$ and $r(a)$ in $End(H)$ by $l(a)(b) = ab$ and $r(a)(b) = ba$.

Proposition 1.9[5]

Suppose that S is the antipode of H. Let Λ be a left integral for H and ω be a right integral for H^* which satisfy $<\Lambda, \omega>=1$. Then

(1) $Tr(r(a) \circ S^2 \circ R(a^*)) = <\omega, a><a^*, \Lambda>$ for all $a \in H$, $a^* \in H^*$.

(2) The functional $\omega_r \in H^*$ defined by $\omega_r(a) = Tr(r(a) \circ S^2)$ for all $a \in H$ is a right integral for H^*.

Proposition 1.10[5]

Suppose that S is the antipode of H. Then the following are equivalent:

(1) H and H^* are semisimple.

(2) $Tr(S^2) \neq 0$.

Proposition 1.11[5]

Suppose that S is the antipode of H.

(1) Let g and α be the distinguished grouplike elements for H and H^* respectively. Then $S^4 = \tau_g \circ (\tau_{\alpha^{-1}})^*$ or equivalently, $S^4(a) = g(\alpha \to a \leftarrow \alpha^{-1})g^{-1}$, for all $a \in H$.

(2) If H and H^* are unimodular , in particular if H and H^* are semisimple, then $S^4 = 1_H$.

(3) $Tr(S^2) = (Dim(H))Tr(S^2|_{x_H} H)$.

Theorem 1.12[6]

Let H be a Hopf algebra over K. Then the following are equivalent:

(1) All left $H-$ comodules are completely reducible.

(2) $<\lambda,1>\neq 0$ for some $\lambda\in\int^{r}$. .
(3) $H=K1\otimes C$ for some subcoalgebra C of H .
(4) $<\lambda,1>\neq 0$for some $\lambda\in\int^{l}$
(5) All right $H-$ comodules are completely reducible.

Theorem 1.13[6]

Let H be a cosemisimple Hopf algebra with antipode S. Then $S^2(C)=C$ for all simple subcoalgebras C of H.

2 The order of the antipode

Lemma 2.1

Suppose $\eta\in G(H^*)$, $g\in G(H)$, $m,n\in H^*$ and $x\in L_\eta$ and $x\in L_\eta$ such that $m\rightarrow x=x\leftarrow n$. Then $m\in L_g$ and $n\in R_g$.
Proof Let $h^*,g^*\in H^*$.Then

$$\begin{aligned}(g^*h^*m)(x)&=\sum(g^*h^*)(x_1m(x_2))\\&=(g^*h^*)(g)\\&=g^*(g)h^*(g)\\&=\sum g^*(m(x_2)x_1)h^*(g)\\&=(g^*h^*(g)m)(x)\end{aligned}$$

which shows that$(g^*(h^*m-h^*(g)m))(x)=0$, so $(h^*m-h^*(g)m)(x\leftarrow H^*)$ .But $x\leftarrow H^*=H$, since $L_\eta\leftarrow\eta=L_\varepsilon$ and $L_\varepsilon\leftarrow H^*=H$ (applied for the dual of H^{op}). This shows that $h^*m=h^*(g)m$, and so $m\in L_g$. The fact that $n\in R_g$ is proved in a similar way.

Corollary 2.2

If $m\in H^*, x\in L_\varepsilon$, and $m\rightarrow x=1$, then $m\in L_1$ and $x\leftarrow m=d$.
Proof If $h^*\in H^*$, then

$$\begin{aligned}h^*(x\leftarrow m)&=\sum h^*(x_2)m(x_1)\\&=(mh^*)(x)\\&=h(d)m(x)\\&h^*(m(x)d)\end{aligned}$$

Applying to the relation $\sum m(x_2)x_1=1$ we get $m(x)=1$. This shows that $x\leftarrow m=d$. The fact that $m\in L_1$ is proved by Lemma 2.1.

Lemma 2.3

Suppose $x\in L_\eta$, $g\in G(H)$, $m\in H^*$ such that $m\rightarrow x=g$.Then for any $h^*\in H^*$ we have $\eta(g)h^*(1)=\sum h^*(x_1)m(gx_2)$.
Proof From the fact that $\Delta(h^*)=\sum h_1^*\otimes h_2^*$, $g=m\rightarrow x$ and $\eta(g)x=gx$,we have ,we have

$$\begin{aligned}\eta(g)h^*(1)&=\sum\eta(g)h_1^*(g^{-1})h_2^*(g)\\&=h_1^*(g^{-1})h_2^*(m(x_2)x_1\eta(g))\\&=h_1^*(g^{-1})h_2^*(m(gx_2)gx_1)\\&=\sum h^*(g^{-1}gx_1)m(gx_2)\\&=\sum h^*(x_1)m(gx_2)\end{aligned}$$

Lemma 2.4

Let Let $x\in L_\eta$, $g\in G(H)$, $m\in H^*$, and $\eta\in G(H^*)$ such that $m\rightarrow x=g$. Then for any $h\in H$ we have $S(g^{-1}(\eta\rightarrow h)\ =(m\leftarrow h)\rightarrow x$.
Proof If $h^*\in H^*$. Then

$$h^*(S(g^{-1}(\eta \to h))) = \sum h^*(S(h_1)g)\eta(h_2)$$
$$= \sum \eta(g)h^*(S(h_1)g)\eta(g^{-1}h_2)$$
$$= \eta(g)((h^*S)\eta)(g^{-1}h)$$
$$= \sum ((h_1^*S)\eta)(g^{-1}h)\eta(g)h_2^*(1)$$
$$= \sum ((h_1^*S)\eta)(g^{-1}h)h_2^*(m(gx_2)x_1)$$
$$= \sum (h_1^*S)(g^{-1}h_1)\eta(g^{-1}h_2)h_2^*(m(gx_2)x_1)$$
$$= \sum h_1^*(S(h_1)g)h_2^*(m(gg^{-1}h_3x_2)g^{-1}h_2x_1)$$
$$= \sum h_1^*(S(h_1)gg^{-1}h_2x_1m(h_3x_2))$$
$$= \sum h^*(x_1m(hx_2))$$
$$= h^*((m \leftarrow h) \to x).$$

Remark 2.5

If we write the formula from Lemma2.4 for the Hopf algebras $H, H^{cop}, H^{op,cop}$ and H^{op}, we get that for any $h \in H$ the following relations hold:

Suppose $x \in L_\eta, m \to x = g,$ then $S(g^{-1}(\eta \to h)) = (m \leftarrow h) \to x;$

Suppose $x \in R_\eta, m \to x = g,$ then $S^{-1}((\eta \to h)g^{-1}) = (h \to m) \to x;$

Suppose $x \in R_\eta, x \leftarrow n = g,$ then $S((h \leftarrow \eta)g^{-1}) = x \leftarrow (h \to n);$

Suppose $x \in L_\eta, x \leftarrow n = g,$ then $S^{-1}(g^{-1}(h \leftarrow \eta)) = x \leftarrow (n \leftarrow h).$

In particular

If $x \in L_\varepsilon, m \to x = 1,$ then $S(h) = (m \leftarrow h) \to x$ (2.1)

If $x \in R_\alpha = L_\varepsilon, m \to x = 1,$ then $S^{-1}(\alpha \to h) = (h \to m) \to x$ (2.2)

If $x \in R_\alpha = L_\varepsilon, x \leftarrow n = d,$ then $S((h \leftarrow \alpha)d^{-1}) = x \leftarrow (h \to n)$ (2.3)

If $x \in L_\varepsilon, x \leftarrow n = g,$ then $S^{-1}(g^{-1}h) = x \leftarrow (n \leftarrow h)$ (2.4)

Theorem 2.6

For any $h \in H$ we have $S^4(h) = d^{-1}(\alpha \to h \leftarrow \alpha^{-1})d$.

Proof Suppose $x \in L_\varepsilon = R_\alpha$, and $m \in H^*$ with $m \to x = 1$. Corollary 2.2 shows that $m \in L_1$ and $m \in L_1$ and $x \leftarrow m = d$. Moreover, we have

$(S^4(h) \to m) \to x = S^{-1}(\alpha \to S^4(h))$ (by (2.2))

$= S^{-1}(S^4(\alpha \to h))$

$= S(S^2(\alpha \to h))$

$= S(S^2(\alpha \to h))$ (by (2.1))

Since the map from H^* to H, sending $h^* \in H^*$ to $h^* \to x \in H$ is bijective, we obtain $S^4(h) \to m = m \leftarrow S^2(\alpha \to h)$

.

On the other hand,

$x \leftarrow (m \leftarrow S^2(\alpha \to h) = S^{-1}(d^{-1}S^2(\alpha \to h)$ (by (2.4))

$= S^{-1}(S^2(d^{-1}(\alpha \to h)))$

$= S(d^{-1}(\alpha \to h)$

$= S(d^{-1}(\alpha \to h)d^{-1})$

$= S(((d^{-1}(\alpha \to h \leftarrow \alpha^{-1})d) \leftarrow \alpha)d^{-1})$

$= x \leftarrow (d^{-1}(\alpha \to h \leftarrow \alpha^{-1})d) \to m)$ (by (2.3))

Since the map $h^* \mapsto (x \leftarrow h^*)$ from H^* to H is bijective, we obtain that

$m \leftarrow S^2(\alpha \to h) = (d^{-1}(\alpha \to h \leftarrow \alpha^{-1})d) \to m$

We got that

$S^4(h)\to m=(d^{-1}(\alpha\to h\leftarrow\alpha^{-1})d)\to m$

then the formula follows from the bijectivity of the map $h\mapsto (h\to m)$ from H to H^*.

Theorem 2.7

Let H be a finite dimensional Hopf algebra. Then the antipode S has finite order.

Proof By Theorem 2.6, we obtain by induction that

$S^{4n}(h)=d^{-n}(\alpha^n\to h\leftarrow\alpha^{-n})d^n$ for any positive integer $n.$ Since $G(H)$ and $G(H^*)$ are finite groups, their elements have finite orders, so there exists p for which $d^p=1$ and $\alpha^p=\varepsilon.$ Then it follows that $S^{4p}=I.$

3 Characterizations of semisimple Hopf algebras

Semisimlpe Hopf algebras are finite-dimsnsional by part (2) of Lemma 1.3. We characterize finite-dimensional Hopf algebras which are semisimple in the algebraically closed characteristic zero case. To this end we calculate a trace.

Lemma3.1

If C is a simple coalgebra over $K,$ and T is a diagonalizable coalgebra automorphism of C. The

$$Tr(T)=(\sum_{i=1}^{n}\lambda_i)(\sum_{i=1}^{n}\lambda_i^{-1})$$

where $\lambda_1,\lambda_2,\cdots,\lambda_n$ are eigenvalues for T.

Proof By lemma1.5 we obtain that $C\cong C_n(K)$ for some $n\geq 1.$ Thus we may assume $C=C_n(K)$. The crux of the proof will be to show that there is a simple left coideal M of C such that $T(M)\subseteq M.$ Necesarily $Dim(M)=n.$

T^* is an algebra automorphism of $C^*=M_n(K).$ By Skolem-Noether Theorem, there is an invertible matrix $u\in M_n(K)$ such that $T^*(a)=uau^{-1}$ for all $a\in M_n(K).$ Identify $C^*=M_n(K)$ with $End(V)$,where V is $n-$ an dimensional vector space over $K.$ Since K is algebraically closed, u has an eigenvalue $\lambda\in K.$ Let $v\in V$ be a non-zero vector satisfying $u(V)=\lambda v.$ Regard $End(V)$ and V as left $End(V)-$ modules via function composition and evaluation respectively. Then V is a simple module and the evaluation map

$$e_v:End(V)\to V \text{ given by } e_v(a)=a(v) \text{ for all } a\in End(V)$$

is a module map. Therefore $L=Ker(e_v)=\{a\in End(V)|a(v)=0\}$ is a maximal left ideal of $End(V)$ of codimension $n^2-n.$ Observe that $T^*(L)\subseteq L.$ Set $M=L^{\perp}.$ Then M is a minimal left coideal of C by Lemma 1.4 and $T(M)\subseteq M$ by Lemma 1.6. and Using Remark1.7 we see that $Dim(M)=n.$

Since T is diagonalizable and $T(M)\subseteq M$ it follows that the restriction $T|M$ is diagonalizable. Let be a basis of eigenvectors for and let $\lambda_1,\cdots,\lambda_n\in K$ satisfy $T(m_i)=\lambda_i m_i$ for all $1\leq i\leq n.$ Then $\lambda_1,\cdots,\lambda_n$ are non-zer scalars since is noe-one. For each $1\leq i\leq n$ write $\Delta(m_i)=\sum_{j=1}^{n}c_{i,j}\otimes m_j.$ Then the $c_{i,j}$'s satisfy the comatrix identities and thus span a non-zero subcoalgebra D of C Since C is simple $D=C.$ Since $Dim(C)=n^2$ necessarily the $c_{i,j}$'s from a basis for C. Applying $T\otimes T$ to both sides of the equation for $\Delta(m_i)$ yields $\sum_{j=1}^{n}\lambda_i c_{i,j}\otimes m_j=\sum_{j=1}^{n}T(c_{i,j})\otimes\lambda_j m_j.$ Therefore $T(c_{i,j})=\lambda_i\lambda_j^{-1}c_{i,j}$ for all $1\leq i,j\leq n.$ Since $\{c_{i,j}\}_{1\leq i,j\leq n}$ is a basis for C we calculat

$$Tr(T)=\sum_{i,j=1}^{n}\lambda_i\lambda_j^{-1}=(\sum_{i=1}^{n}\lambda_i)(\sum_{i=1}^{n}\lambda_i^{-1}).$$

Theorem 3.2

Let H be a Hopf algebra with antipode S over K. Then the following are equivalent.

(1) H is cosemisimple.

(2) $Tr(S^2)\neq 0.$

(3) H is semisimple.

(4) $S^2=1_H.$

(5) $\omega:H\to K$ defined by $\omega(a)=Tr(r(a))$ for all $a\in H$ is a right integral for $H.$

Proof (1) $\Longrightarrow$ (2). Since H is cosemisimple it is the direct sum of its simple subcoalgebras . Let C be a simple subcoalgebra of H. Then $S(C)=C$ By Theorem1.13 . Now S^2 has finite order by part (1) of Theorem Proposition1.11. Since K is algebraically closed of characteristic zero S^2 is diagonalizable. Thus $Tr(S^2)=(\sum_{i=1}^{n}\lambda_i)(\sum_{i=1}^{n}\lambda_i^{-1})$ where $\lambda_1,\cdots,\lambda_n$ are roots of unity by Lemma 3.1. Since the characteristic of K is zero we may assume that $\lambda_1,\cdots,\lambda_n\in$ C, the field of compex mubers. Thus

$$Tr(S^2|C)=(\sum_{i=1}^{n}\lambda_i)(\sum_{i=1}^{n}\lambda_i^{-1})=(\sum_{i=1}^{n}\lambda_i)(\overline{\sum_{i=1}^{n}\lambda_i})=\left|\sum_{i=1}^{n}\lambda_i\right|^2$$

is a non-negative real number. Therefor $Tr(S^2)=1+\sum_{C}Tr(S^2|C)\geq 1,$ ewhere C runs over the simple subcoalgebras $C\neq K1$ of H. We have shown that $Tr(S^2)\neq 0$. .

(2)⟹(3). It is pretty obvious by Proposition1.10.

(3)⟹(4). Assume that H is semisimple. Then H^* is cosemisimple. We have just show H^* is semisimple; thus H is semisimple and cosemisimple. In particular $Tr(S^2)\neq 0$. Now $Tr(S^2)=(Dim(H))Tr(S^2|_{x_H} H)$ by part (3) of Proposition 1.11 and $S^4=1_H$ by part (2) of Porposition1.11. Since the characteristic of K is not 2, the last equation implies S^2 is a diagonalizable endomorphism of H with eigenvalues ± 1. Choose a basis of eigenvectors for S^2. Let n_+ be the number of basis vectors belonging to the eigenvalue 1 and let $n\_$ be the number belonging to -1. By the preceding trace formula $n_+ - n_- = (n_+ + n_-)m$ for some integer m which is not zero since $Tr(S^2)\neq 0$. Squaring both sides of this equation yields $-2n_+n_- = (m^2-1)n_+^2 + 2m^2n_+n_- + (m^2-1)n_-^2 \geq 0$. Therefore $n_+n_- = 0$. Since $n_+ \neq 0$ necessarily $n_- = 0$. We have shown $S^2 = 1_{H.}$

(4)⟹(5). That it is very simple follows by part (2) of Proposition1.9.

(5)⟹(1). Since $\omega(1) = Dim(H)1 \neq 0$, thus our proof is complete by Theorem1.12.

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