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# Asymptotic Behaviors of Heat Equation Associated with Symmetric a-stable-like Process

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**Abstract:** This paper is devoted to asymptotic behaviors of heat equation corresponding to symmetric $\alpha-\text{stable}-\text{like}$ process $X:=(X_t)_{t>0}$ on metric measure space.Denote $p_t(x,y)$ the heat kernel of the process,and by $u(t,x)$ the solution of the associated heat equation.We then establish asymptotic behaviors between $p_t(x,y)$ and $u(t,x)$,which enjoy similar properties of these in Riemannian manifolds.

**Keywords:** a-stable-like process;Heat kernel;Heat equation;Asymptotic behaviors

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## Introduction

In the theory of partial differential equations,the study of heat equations has a long history.Meanwhile,with the tremendous progress of mathematical theory over the past two centuries,the study of heat equations has developed rapidly.Specially,the asymptotic properties of heat equation solutions have attracted great attention from many scholars.For example,in ref.[1]the asymptotic properties of classical heat equation solutions in Euclidean space were comprehensively discussed and these properties hold true for all integrable initial values $u_0(x)$.And under some additional assumptions,the accuracy of the results is verified.At the same time,ref.[2]pointed out that the long-term asymptotic properties of the solution of the heat equation in hyperbolic space $\mathrm{H_n}$ are not valid for all initial values $u_0(x)\in L^1(\mathrm{H_n})$.Ref.[3]studied the long-term asymptotic properties of the solution of the heat equation in noncompact symmetric spaces by using harmonic analysis methods.Recently,Professor Grigor'yan et al.studied asymptotic properties of the solution of the heat equation with initial value $L^1$ corresponding to Brownian motion in ref.[4]

## 1. Main Conclusions

Before giving the main conclusions of this paper,we first review the relevant conclusions of the ref.[5]mentioned above.

Theorem 1.1.Let $M$ be a complete,connected,non-compact Riemannian manifold with non-negative Ricci curvature.Let $u_0(x)\in L^1(M,\mu)$.Then the solution of the heat equation satisfies for any $x_0\in M$,

$$\lim_{t\to\infty}\|u(t,\cdot)-M^*h_t(\cdot,x_0)\|_{L^1(M,\mu)}=0,$$

and $$\lim_{t\to\infty}\||u(t,\cdot)-M^*h_t(\cdot,x_0)|V(\cdot,\sqrt{t})\|_{L^\infty(M,\mu)}=0,$$

with $M^*:=\int_M u_0(x)\mu(\mathrm{d}x)$.Heat kernel $h_t(x,y)$ is positive fundamentals of the heat equation.

Inspired by ref.[4],we naturally raise a question:If we consider the symmetric $\alpha-\text{stable}$ process $X:=(X_t)_{t\geq 0}$ on the metric space $M$,does the corresponding solution of the heat equation still have similar properties?The purpose of this paper is to give a positive answer to the above questions.To illustrate our results,we first give the framework and main conditions.

Let $(M, d, \mu)$ be the metric measure space, where $\mu$ is a fully supported non-negative Radon measure on $M$. Let $B(x, r)$ represent the opening ball with $x$ as the center and $r$ as the radius on $(M, d)$, i.e.

$$B(x,r) := \{y \in M: d(x,y) < r\}$$

Let $V(x,r) = \mu(B(x,r))$. In this paper, we assume that $M$ satifies the volume multiplication condition, i.e.

(VD):(Volume doubling condition)There exists a positive constant $C_1$. For all $x \in M, r > 0$, we have

$$V(x,2r) \le C_1 V(x,r). \tag{1.1}$$

(RVD):(Reverse volume doubling condition)There exist positive constants $c, C, v, v'$. For all $x \in M$ and $0 < r \le R$, we have

$$c\left(\frac{R}{r}\right)^{v'} \le \frac{V(x,R)}{V(x,r)} \le C\left(\frac{R}{r}\right)^{v}. \tag{1.2}$$

We notice that if $M$ is connected and unbounded, we can get(VD)$\rightarrow$(RVD). See ref.[5,inference 5.3].

This paper will consider the symmetric Markov process $(X_t)_{t\ge 0}$ on $L^2(M,\mu)$, where $(P_t)_{t\ge 0}$ is the corresponding Markov semigroup of $(X_t)_{t\ge 0}$. We assume that $(X_t)_{t\ge 0}$ has a heat kernel $p_t(x, y)$, i.e.

$$P_t f(x) = \int_M f(y) p_t(x,y)\mu(\mathrm{d}y), f(x) \in L^2(M,\mu).$$

Among them, $L^p(M,\mu)$ represents the $L^p$ space on $M$.

We assume that $p_t(x, y)$ satisfies the following condition, i.e.

(1)heat kernel upper bound:

$$p_t(x,y) \lesssim \left(\frac{1}{V(x,\phi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi(d(x,y))}\right). \tag{1.3}$$

(2)$\theta$-Hölder continuity:if there is $\theta \in (0,1]$, we have

$$|p_t(x,y) - p_t(x,x_0)| \lesssim \left(\frac{d(x_0,y)}{\phi^{-1}(t)}\right)^{\theta} [\, p_t(x,y) + p_t(x,x_0)]. \tag{1.4}$$

Let's note here that $\mathbb{R}_+ := [0,+\infty)$ and $\phi$ is a strictly monotonic increasing function on $\mathbb{R}_+$ satisfying $\phi(0) = 0$ and $\phi(1) = 1$. And there is a constant $c_1, c_2 > 0, \beta_1 \ge \beta_2 > 0$. For all $0 < r \le R$, there are the following scaling conditions:

$$c_1\left(\frac{R}{r}\right)^{\beta_1} \le \frac{\phi(R)}{\phi(r)} \le c_2\left(\frac{R}{r}\right)^{\beta_2}. \tag{1.5}$$

Notation $A \lesssim B$ is equivalent to the existence of a constant $C$ such that $A \le CB$.

Let $\mathcal{L}$ be the infinitesimal generator operator of semigroup $(P_t)_{t\ge 0}$. Consider the following heat equation

$$\begin{cases} \partial_t u = \mathcal{L}u, \\ u(0,x) = u_0(x). \end{cases} \tag{1.6}$$

It's easy to know if the initial value $u_0 \in L^P(M,\mu)$, the heat equation has a unique solution $u(t,\cdot) \in L^P(M,\mu)$. And it can be represented as

$$u(t,x) = \int_M p_t(x,y) u_0(y)\mu(\mathrm{d}y).$$

The main theorems of this article are as follows:

Theorem 1.2. Under above assumptions, we continue the following content.

Let $u_0 \in L^P(M,\mu)$ and note $M_* := \int_M u_0(x)\mu(\mathrm{d}x)$. The solution of the heat equation satisfies for any $x_0 \in M$,

$$\lim_{t\to\infty} \|u(t,\cdot) - M_* p_t(\cdot, x_0)\|_{L^1(M,\mu)} = 0, \tag{1.7}$$

and
$$\lim_{t\to\infty} \||u(t,\cdot) - M_* p_t(\cdot, x_0)|V(\cdot, \phi^{-1}(t))\|_{L^\infty(M,\mu)} = 0 \tag{1.8}$$

Remark 1.3.(1)Combining with theorem 1.2 and the interpolation theorem,it can be obtained when $u_0 \in L^p(M,\mu)$with $1 < p < \infty$.For any $x_0 \in M$,we can deduce that

$$\lim_{t\to\infty} \||u(t,\cdot) - M_* p_t(\cdot, x_0)|V(\cdot, \phi^{-1}(t))^{1/p'}\|_{L^p(M,\mu)} = 0.$$

here $p' > 1$ satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

(2)Comparing with Theorem 1.1,it can be seen that for symmetric $\alpha - \text{stable}$ processes in metric measure spaces,their corresponding heat equations have the same asymptotic properties as those on Riemannian manifolds.

## 2. Proof of Theorem

In this section,we will prove Theorem 1.2 in three steps.Firstly,by using the initial value of continuous compact support,we will prove that the solution of the heat equation(1.6)converges in $L^1$.Next,we use density to transition special initial values to all $L^1$ initial values to prove that the conclusion still holds.Finally,transition to $L^\infty$ convergence by the same method.To sum up,we will provide a complete proof of Theorem 1.2.

Before the formal proof of Theorem 1.2,we will provide two lemmas,which will be frequently used in the proof process of Theorem 1.2.

Lemma 2.1.Under the conditions of(1.3),for any $x_0 \in M$and $t > 0$,we obtain

$$\int_M p_t(x_0, x)\mu(dx) \lesssim 1. \tag{2.1}$$

And when $t \le \phi(r)$,we obtain

$$\int_{B(x_0,r)^c} p_t(x_0,x)\mu(dx) \lesssim \frac{t}{\phi(r)}. \tag{2.2}$$

Proof:We first prove equation(2.2).For any $t \le \phi(r)$according to equation(1.3),we obtain

$$\int_{d(x,x_0)\ge r} p_t(x,x_0)\mu(dx) \lesssim \sum_{j=0}^{\infty} \int_{2^j r \le d((x,x_0)<2^{j+1}r} \frac{t}{V(x_0, d(x,x_0))\phi(d(x,x_0))}\mu(dx)$$

$$\lesssim \sum_{j=0}^{\infty} \frac{V(x_0, 2^{j+1}r)}{V(x_0, 2^j r)} \cdot \frac{t}{\phi(2^j r)} \lesssim 2^v \sum_{j=0}^{\infty} \frac{t}{\phi(2^j r)}.$$

Among them,the last inequality is obtained according to equation(1.2).Then by using equation(1.5)to estimate $\phi(2^j r)$,it can be concluded that

$$2^v \sum_{j=0}^{\infty} \frac{t}{\phi(2^j r)} \le \frac{2^v}{c_1} \cdot \frac{t}{\phi(r)} \cdot \sum_{j=0}^{\infty} \frac{1}{2^{\beta_1 j}}.$$

Obviously,equation(2.2)holds.

To prove the validity of equation(2.1),we assume that $r = \phi^{-1}(t)$,then we have

$$\int_M p_t(x,x_0)\mu(dx) \lesssim \int_M \left(\frac{1}{V(x_0, \phi^{-1}(t))} \wedge \frac{t}{V(x_0, d(x,x_0))\phi(d(x,x_0))}\right)\mu(dx)$$

$$= \int_{B(x_0,r)^c} \frac{t}{V(x_0, d(x,x_0))\phi(d(x,x_0))}\mu(\mathrm{d}x)$$

$$+ \int_{B(x_0,\phi^{-1}(t))} \frac{1}{V(x_0,\phi^{-1}(t))}\mu(\mathrm{d}x)$$

$$\lesssim 1 + \int_{B(x_0,\phi^{-1}(t))} \frac{1}{V(x_0,\phi^{-1}(t))}\mu(\mathrm{d}x) = 2.$$

The proof of equation(2.1)is completed.

Note $\varphi(t)$a definite positive function and when it satisfies $t \to \infty$, $\varphi(t) \to 0$.For $x_0 \in M$,define

$$A_t = \left\{x \in M: \varphi(t)\phi^{-1}(t) \le d(x,x_0) \le \frac{\phi^{-1}(t)}{\varphi(t)}\right\}. \tag{2.3}$$

Lemma 2.2.Under the conditions of(1.3),for sufficiently large $t$,we have

$$\int_{M\setminus A_t} p_t(x,x_0)\,\mu(\mathrm{d}x) \lesssim \varphi(t)^{v'\wedge\beta_1}, \tag{2.4}$$

Among them,constants $v'$and $\beta_1$are respectively taken from equations(1.2)and(1.5).Specifically,when $t \to \infty$,we can obtain

$$\int_{A_t} p_t(x,x_0)\mu(\mathrm{d}x) \to 1. \tag{2.5}$$

Proof:We first prove that equation(2.4)holds.Note that

$$M \setminus A_t = B(x_0,\varphi(t)\phi^{-1}(t)) \cup B(x_0,\phi^{-1}(t)/\varphi(t))^c.$$

We estimate the integration ranges of $B(x_0,\varphi(t)\phi^{-1}(t))$and $B(x_0,\phi^{-1}(t)/\varphi(t))^c$

separately.Assume that $t$is large enough and make $\varphi(t) < 1$.By using equations(1.2)and(1.3),we have

$$\int_{d(x,x_0)<\varphi(t)\phi^{-1}(t)} p_t(x,x_0)\mu(\mathrm{d}x) \lesssim \int_{d(x,x_0)<\varphi(t)\phi^{-1}(t)} \frac{1}{V(x_0,\phi^{-1}(t))}\mu(\mathrm{d}x)$$

$$\lesssim \frac{V(x_0,\varphi(t)\phi^{-1}(t))}{V(x_0,\phi^{-1}(t))} \lesssim \varphi(t)^{v'}.$$

On the other hand,let $r = \frac{\phi^{-1}(t)}{\varphi(t)}$ that is there is $t = \phi(r\varphi(t))$.Therefore,according to equation(2.2),we have

$$\int_{d(x,x_0)\ge\frac{\phi^{-1}(t)}{\varphi(t)}} p_t(x,x_0)\mu(\mathrm{d}x) \lesssim \int_{B(x_0,r)^c} \frac{t}{V(x_0,d(x,x_0))\phi(d(x,x_0))}\mu(\mathrm{d}x)$$

$$\lesssim \frac{t}{\phi(r)} \lesssim \frac{\phi(r\varphi(t))}{\phi(r)} \lesssim \varphi(t)^{\beta_1}.$$

Combining the above two parts,we can obtain

$$\int_{M\setminus A_t} p_t(x,x_0)\mu(\mathrm{d}x) = \int_{d(x,x_0)<\varphi(t)\phi^{-1}(t)} p_t(x,x_0)\mu(\mathrm{d}x) + \int_{d(x,x_0)>\frac{\phi^{-1}(t)}{\varphi(t)}} p_t(x,x_0)\mu(\mathrm{d}x)$$

$$\lesssim \varphi(t)^{v'\wedge\beta_1}.$$

The proof of equation(2.4)is completed.

From equation(2.4),it is evident that(2.5)holds when $t \to \infty$.

Let's begin to prove the special case of Theorem 1.2.

Proposition 2.3.Assume(1.3)and(1.4)conditions hold.Let $u_0 \in C_c^{\infty}(M)$satisfying supp $u_0 \subset B(x_0,a)$with$x_0 \in M$, $a>0$.Let $\varphi(t)$be a positive function satisfying when $t \to \infty, \varphi(t) \to 0$ and $\varphi(t)\phi^{-1}(t) \to 0$.When $t$is large enough,the solution of the heat equation(1.6)satisfies

$$||u(t,\cdot)-M_* p_t(\cdot\,,x_0)||_{L^1(M\backslash A_t)} \lesssim \varphi(t)^{v'\wedge\beta_1}, \tag{2.6}$$

and

$$||u(t,\cdot)-M_* p_t(\cdot\,,x_0)||_{L^1(A_t)} \lesssim [\phi^{-1}(t)]^{-\theta}. \tag{2.7}$$

And $A_t$is the annulus defined by equation(2.3).Besides,constant $v'$and $\beta_1$are taken from equations(1.2)and(1.5)respectively and$M_* := \int_M u_0(x)\mu(\mathrm{d}x)$.Specially,when $t$is big enough,

$$||u(t,\cdot)-M_* p_t(\cdot\,,x_0)||_{L^1(M)} \lesssim [\phi^{-1}(t)]^{-\lambda}, \tag{2.8}$$

with $0<\lambda<\min\left(v'\wedge\beta_1,\theta\right)$.

Proof:We first prove that equation(2.6)holds.According to Lemma 2.2,we have

$$||p_t(\cdot\,,x_0)||_{L^1(M\backslash A_t)} \lesssim \varphi(t)^{v'\wedge\beta_1}. \tag{2.9}$$

Therefore we only need to prove that

$$||u(t,\cdot)||_{L^1(M\backslash A_t)} \lesssim \varphi(t)^{v'\wedge\beta_1}. \tag{2.10}$$

By using the Fubini theorem and supp $u_0 \subset B(x_0,a)$,we have

$$||u(t,\cdot)||_{L^1(M\backslash A_t)} = \int_{M\backslash A_t} |u(t,\cdot)|\,\mu(\mathrm{d}x)$$

$$= \int_{M\backslash A_t} |\int_{B(x_0,a)} p_t(x,y)u_0(y))\mu(\mathrm{d}y)|\mu(\mathrm{d}x)$$

$$\leq \int_{B(x_0,a)} |u_0(y)| \left\{\int_{M\backslash A_t} p_t(x,y)\,\mu(\mathrm{d}x)\right\} \mu(\mathrm{d}y).$$

Notice that when $t$is large enough for any$x \in M \setminus A_t$and $y \in B(x_0,a)$,then we have $x \in M \setminus \widetilde{A_{t,y}}$with

$$\widetilde{A_{t,y}} = \left\{x \in M | 2\varphi(t)\phi^{-1}(t) \leq d(x,y) \leq \frac{1}{2}\frac{\phi^{-1}(t)}{\varphi(t)}\right\}.$$

In fact,when $t$is big enough and $x \in M \setminus \widetilde{A_{t,y}}$,then we have

$$d(x,x_0) \leq d(x,y)+d(y,x_0) \leq \frac{1}{2}\frac{\phi^{-1}(t)}{\varphi(t)} + a \leq \frac{\phi^{-1}(t)}{\varphi(t)},$$

and

$$d(x,x_0) \geq d(x,y)-d(y,x_0) \geq 2\varphi(t)\phi^{-1}(t) - a \geq \varphi(t)\phi^{-1}(t).$$

Thus $x \in A_t$.In this way,from Lemma 2.2,it can be concluded that

$$\int_{M\setminus A_t} p_t(x,y)\mu(\mathrm{d}x) \le \int_{M\setminus \widetilde{A_{t,y}}} p_t(x,y)\mu(\mathrm{d}x) \lesssim \varphi(t)^{v'\wedge\beta_1}.$$

The proof of equation(2.10)is completed.Combining equations(2.9)and(2.10),it can be concluded that equation(2.6)holds.

Let's continue to prove that equation(2.7)holds.Firstly

$$u(t,x) - M_* p_t(x,x_0) = \int_M p_t(x,y)u_0(y)\,\mu(\mathrm{d}y) - \int_M p_t(x,x_0)u_0(y)\,\mu(\mathrm{d}y)$$

$$= \int_M u_0(y)\big(p_t(x,y) - p_t(x,x_0)\big)\,\mu(\mathrm{d}y)$$

$$= \int_{B(x_0,a)} u_0(y)\big(p_t(x,y) - p_t(x,x_0)\big)\,\mu(\mathrm{d}y)$$

According to equation(1.4),it can be obtained that

$$|u(t,x) - M_* p_t(x,x_0)| \le \int_{B(x_0,a)} |u_0(y)||p_t(x,y) - p_t(x,x_0)|\,\mu(\mathrm{d}y)$$

$$\lesssim \int_{B(x_0,a)} |u_0(y)| \left(\frac{d(x_0,y)}{\phi^{-1}(t)}\right)^{\theta} [p_t(x,y) + p_t(x,x_0)]\,\mu(\mathrm{d}y).$$

By using the Fubini theorem and $d(x_0,y) \le a$,we have

$$||u(t,x) - M_* p_t(x,x_0)||_{L^1(A_t)} = \int_{A_t} |u(t,x) - M_* p_t(x,x_0)|\,\mu(\mathrm{d}x)$$

$$\lesssim \int_{A_t}\int_{B(x_0,a)} |u_0(y)| \left(\frac{d(x_0,y)}{\phi^{-1}(t)}\right)^{\theta} [p_t(x,y) + p_t(x,x_0)]\,\mu(\mathrm{d}y)\mu(\mathrm{d}x)$$

$$\lesssim [\phi^{-1}(t)]^{-\theta} \int_{B(x_0,a)} |u_0(y)| \left\{\int_{A_t} p_t(x,y)\mu(\mathrm{d}x)\right\} \mu(\mathrm{d}y)$$

$$+ [\phi^{-1}(t)]^{-\theta} \int_{B(x_0,a)} |u_0(y)| \left\{\int_{A_t} p_t(x,x_0)\,\mu(\mathrm{d}x)\right\} \mu(\mathrm{d}y)$$

$$\lesssim [\phi^{-1}(t)]^{-\theta}.$$

The proof of equation(2.7)is completed.

Take $\varepsilon$small enough and let $\varphi(t) = [\phi^{-1}(t)]^{-1}t^{\varepsilon}$.Then combine with equations(2.6)and(2.7),we have

$$||u(t,\cdot) - M_* p_t(\cdot\,,x_0)||_{L^1(M)} \lesssim [\phi^{-1}(t)]^{-\lambda},$$

with $0 < \lambda < \min\left(v' \wedge \beta_1, \theta\right)$.The proof of equation(2.8)is completed.

Based on Proposition 2.3,we now prove the general case of Theorem 1.2

Proof:Based on density,for any fixed $u_0 \in L^1(M,\mu)$and any $\varepsilon > 0$,there is $\widetilde{u_0} \in C_c(M)$to make

$$||u_0 - \widetilde{u_0}||_{L^1(M)} < \frac{\varepsilon}{3}.$$

We first prove that equation(1.7)holds.Let $\widetilde{M_*} = \int_M \widetilde{u_0}(y)\,\mu(\mathrm{d}y)$.We have

$$\left|M_* - \widetilde{M_*}\right| \le \int_M |u_0(y) - \widetilde{u_0}(y)|\,\mu(\mathrm{d}y) = ||u_0 - \widetilde{u_0}||_{L^1(M)} < \frac{\varepsilon}{3}.$$

Thus we conclude

$$||M_* p_t(\cdot\,,x_0) - \widetilde{M_*} p_t(\cdot\,,x_0)||_{L^1(M)} \le \left|M_* - \widetilde{M_*}\right| ||p_t(\cdot\,,x_0)||_{L^1(M)} < \frac{\varepsilon}{3}. \tag{2.11}$$

Let $\widetilde{u}(t,x)$be the solution of the heat equation(1.6)with initial value $\widetilde{u_0}$and we have

$$||u(t,\cdot)-\tilde{u}(t,\cdot)||_{L^1(M)} \le \int_M |u_0(y)-\widetilde{u_0}(y)| \{\int_M p_t(x,y)\,\mu(dx)\}\,\mu(dy) < \frac{\varepsilon}{3}. \tag{2.12}$$

On the other hand,according to equation(2.8)in Proposition 2.3,for a sufficiently large $t$,we have

$$||\tilde{u}(t,\cdot)-\widetilde{M}_* p_t(\cdot\,,x_0)||_{L^1(M)} < \frac{\varepsilon}{3}. \tag{2.13}$$

Combining equations(2.11),(2.12),and(2.13)together,we can obtain

$$||u(t,\cdot)-M_* p_t(\cdot\,,x_0)||_{L^1(M)} < \varepsilon.$$

The proof of equation(1.7)is completed.

Next,we prove that equation(1.8)holds.Since $\widetilde{u_0}$is tightly supported and when $t$is large enough,by using the proof of equation(2.7)in Proposition 2.3,we have

$$\left|\tilde{u}(t,x)-\widetilde{M}_* p_t(x,x_0)\right| \lesssim [\phi^{-1}(t)]^{-\theta}\int_{B(x_0,a)} |u_0(y)| p_t(x,y)\mu(\mathrm{d}y)$$

$$+[\phi^{-1}(t)]^{-\theta}\int_{B(x_0,a)} |u_0(y)| p_t(x,x_0)\,\mu(\mathrm{d}y)$$

$$\lesssim [\phi^{-1}(t)]^{-\theta}\int_{B(x_0,a)} |u_0(y)| \frac{1}{V(x,\phi^{-1}(t))}\mu(\mathrm{d}y)$$

$$\lesssim \frac{\varepsilon}{V\left(x,\phi^{-1}(t)\right)}. \tag{2.14}$$

Furthermore,we have

$$|u(t,x)-\tilde{u}(t,x)| \le \int_M p_t(x,y)\,|u_0(y)-\widetilde{u_0}(y)|\mu(\mathrm{d}y)$$

$$\le \frac{||u_0-\widetilde{u_0}||_{L^1(M)}}{V(x,\phi^{-1}(t))}$$

$$\lesssim \frac{\varepsilon}{V\left(x,\phi^{-1}(t)\right)}. \tag{2.15}$$

and

$$||M_* p_t(x,x_0)-\widetilde{M}_* p_t(x,x_0)|| = \left|M_*-\widetilde{M}\right| p_t(x,x_0) \lesssim \frac{\varepsilon}{V\left(x,\phi^{-1}(t)\right)}. \tag{2.16}$$

Therefore,by combining equations(2.14),(2.15)and(2.16),it can be concluded that equation(1.8)holds.

## 3. Example

Ref.[6]provides a series of equivalent characterizations of the upper bound estimation of the heat kernel corresponding to symmetric $\alpha-\text{stable}$ mixed type processes in metric measure spaces.On the other hand,equation(1.4)requires the heat kernel to satisfy local Hölder continuity,but it cannot be directly obtained from the Hölder continuity of the parabolic harmonic function corresponding to symmetric $\alpha-\text{stable}$ mixed type processes.To be specific,ref.[7]considers the continuity of the heat kernel as follows:

$$|p_t(x,y)-p_t(x,x_0)| \lesssim \left(\frac{d(x_0,y)}{\phi^{-1}(t)}\right)^{\theta}\cdot\frac{1}{V\left(x,\phi^{-1}(t)\right)}. \tag{3.1}$$

Obviously,equation(3.1)is weaker than equation(1.4)

Next,we will provide a simple example to illustrate the assumption that symmetric $\alpha-\mathbf{stable}$ process in Euclidean space satisfies Theorem 1.2.

Let $X:=(X_t)_{t\geq 0}$is a symmetric $\alpha-\mathbf{stable}-\mathbf{Lévy}$ process on $\mathbb{R}^d$,and its corresponding infinitesimal generator is $(\Delta)^{\alpha/2}=-(-\Delta)^{\alpha/2}$.Let $p_t(x,y)$be the corresponding heat kernel of $X$.As is well known,

$$p_t(x,y)=p_t(x-y)\asymp t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}},\quad x,y\in\mathbb{R}^d,t\geq 0, \tag{3.2}$$

and

$$\nabla_x p_t(x)\asymp t^{-1/\alpha}p_t(x),\quad x\in\mathbb{R}^d,t\geq 0.$$

The notation $A\asymp B$means that there exist constants $c_1,c_2>0$ to make $c_1B\leq A\leq c_2B$.Thus,for any $x,y,x_0\in\mathbb{R}^d$ ,there exists $\xi\in(0,1)$and

$$|p_t(x,y)-p_t(x,x_0)|\leq\left|\nabla_x p_t(x,\xi y+(1-\xi)x_0)\right||y-x_0|$$

$$\leq\frac{|y-x_0|}{t^{1/\alpha}}|p_t(x,\xi y+(1-\xi)x_0)|$$

$$\leq\frac{c|y-x_0|}{t^{1/\alpha}}[p_t(x,y)+p_t(x,x_0)].$$

The final step is obtained according to equation(3.2).

In summary,Theorem 1.2 holds for symmetric $\alpha-\mathbf{stable}$ process in Euclidean space.Specifically,it can be obtained from equation(1.8)that

$$\lim_{t\to\infty}\left[|u(t,\cdot)-M_*p_t(\cdot,x_0)|t^{-d/\alpha}\right]=0.$$

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